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AN ELLIPSOIDAL DROP OR VORTEX IN AN INHOMOGENEOUS FLOW

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All motions of an ideal incompressible fluid with a piecewise constant density and vorticity which are discontinuous on the surface of an ellipsoid are found. A tangential discontinuity on the ellipsoid and linear growth in the liquid velocity at infinity are allowed.

1. Formulation of the Problem. In the well-known Kirchhoff elliptical vortex [1] the fluid has a continuous velocity and rests at infinity. It is assumed in the present paper that the vortex is piecewise-constant, and the vorticity- or density-discontinuity surfaces are ellipsoids (ellipses). However, a tangential velocity discontinuity on the ellipsoid and linear growth in the velocity at infinity are allowed. Nontrivial spatial solutions exist for such a generalized formulation. All such solutions are found in this work. This class of solutions contains, in particular, all well-known generalizations of the Kirchhoff vortex [2–4] (all of them are plane). Our generalization has a physical meaning [5] and is also of interest in connection with Lavrent'ev's turbulence model [6]. For brevity, plane solutions are not presented.

We consider the motion of an ideal incompressible fluid which has density ρ_0 inside ellipsoid S and a unit density outside ellipsoid S. Let the vortex $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ be independent of the point $\mathbf{x} = (x_1, x_2, x_3)$ in space both inside and outside S but dependent on the time t and equal to ω_0 and ω_1, respectively. The following notation is used: $\mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$ is the fluid velocity, $p(\mathbf{x}, t)$ is the pressure, and $\nabla = (\partial/\partial x_1,\ \partial/\partial x_2,\ \partial/\partial x_3)$.

All variables are dimensionless. There must be no external forces in a drop ($\rho_0 \neq 1$). In the case of a vortex ($\rho_0 = 1$), the potential mass forces are included in the pressure and has no effect on the motion. Therefore, it is assumed that external forces are absent. The coordinate origin is placed at the center of the ellipsoid, which is assumed to move without acceleration. Then the equation of the surface of S is $f \equiv \mathbf{x} \cdot A\mathbf{x} - 1 = 0$, where A is a symmetric, positively defined matrix. The Euler equations are valid for the fluid. The nonpenetration condition must be held both inside and outside S:

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0 \qquad \text{for} \quad f = 0, \tag{1.1}$$

and the pressure must be continuous. Capillary forces are disregarded. At infinity we have $\mathbf{u}(\mathbf{x}, t) = O(|\mathbf{x}|)$, where $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the vector length.

2. Equations of Motion.

Theorem.

$$\mathbf{u} = \begin{cases} B_0\mathbf{x} & \textit{inside} \quad S, \\ B_1\mathbf{x} & \textit{outside} \quad S, \end{cases}$$

where B_0 and B_1 are functions of t.

The proof is omitted.

We substitute the function $\mathbf{u}$ into the equation of incompressibility:

$$\mathrm{sp}B_0 \equiv \sum_{i=1}^{3} B_{0ii} = 0, \qquad \mathrm{sp}B_1 = 0. \tag{2.1}$$

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Substituting the expression for $\mathbf{u}$ into the equation of fluid motion inside S, we have

$$\dot{B}_0 + B_0^2 = (\dot{B}_0 + B_0^2)^*, \qquad p = -\frac{\rho_0}{2}\mathbf{x}\cdot(\dot{B}_0 + B_0^2)\mathbf{x} + \frac{q_0}{2}.$$

Here, the dot and the asterisk denote a derivative with respect to time and a conjugate matrix, respectively; $q_0(t)$ is an arbitrary function. Similar equations hold outside S with an arbitrary function $q(t)$. Since the pressure is defined with accuracy to the addition of an arbitrary function of time, one can let $q_0 = 0$. The limiting values of $\mathbf{u}$ on S both inside and outside the ellipsoid are then substituted into the nonpenetration condition (1.1) and the pressure-continuity condition on S. We recast the resulting system of equations as follows:

$$\begin{gathered}\rho_0(\dot{B}_0 + B_0^2) - \dot{B}_1 - B_1^2 + qA = 0, \qquad \dot{B}_0 + B_0^2 - (\dot{B}_0 + B_0^2)^* = 0,\\ \dot{A} + AB_0 + B_0^*A = 0, \qquad (B_0 - B_1)A^{-1} + A^{-1}(B_0 - B_1)^* = 0,\end{gathered} \tag{2.2}$$

where $q = \mathrm{sp}(-\rho_0 B_0^2 + B_1^2)/\mathrm{sp}A$. The left-hand sides of the first and the last (after multiplication by A) equations have zero traces by virtue of (2.1). Therefore, system (2.2) contains 22 independent equations for 22 unknowns A, B_0, and B_1 on the manifold (2.1).

For $\rho_0 = 1$, the problem has the nontrivial solution $B_0 = B_1$. Ellipsoid S is a liquid surface in this case. For $\rho_0 \neq 1$ and $B_0 = B_1$, system (2.2) has the same form as the equations of motion of a single ellipsoid [7–10] and has energy, circulation, and angular moment integrals. Below, we consider only the case $B_0 \neq B_1$, wherein analogs of the above-mentioned integrals are available.

The last equation of (2.2) signifies that matrix $F = (B_0 - B_1)A^{-1}$ is antisymmetric. We introduce F as an unknown function and exclude matrix $B_1 = B_0 - FA$ from the first equation. The resulting equation is then multiplied by A^{-1} on the right and $\dot{A}$ is excluded

$$(\rho_0 - 1)(\dot{B}_0 + B_0^2)A^{-1} + \dot{F} + B_0F - FB_0^* - FAF + qI = 0,$$

where I is an identity matrix. We separate the symmetric and antisymmetric parts taking into account the second equation of (2.2). If $\rho_0 = 1$, we have

$$\dot{F} = 0; \tag{2.3}$$

$$G \equiv B_0F - FB_0^* - FAF + qI = 0. \tag{2.4}$$

If $\rho_0 \neq 1$, the symmetric part is solved uniquely with respect to $\dot{B}_0 + B_0^2$:

$$\frac{1}{2}(\rho_0 - 1)(\dot{B}_0 + B_0^2) = H(A, G),$$

where H is a matrix function of the stated arguments. This assertion is obvious in the coordinate system in which the matrix A is diagonal. System (2.2) is reduced to the normal form of order of 17 on the manifold $\mathrm{sp}B_0 = 0$:

$$\dot{F} + HA^{-1} - A^{-1}H = 0, \qquad \dot{B}_0 + B_0^2 = \frac{2}{\rho_0 - 1}H, \qquad \dot{A} + AB_0 + B_0^*A = 0.$$

3. Particular Solution: Rigidly Rotating Ellipsoid. System (2.2) has a solution that describes the rigid rotation of ellipsoid S around the x_3 axis with constant angular velocity ω_0. We write the solution in the coordinate system rotating together with the ellipsoid.

For $\rho_0 = 1$, we have

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$

The relative fluid velocity is $\mathbf{u} = (2\omega_0/A_{11})(A_{11}x_2 + A_{23}x_3,\ -A_{11}x_1,\ 0)$ (it equals zero inside S). The flow is steady in this coordinate system.

When $\rho_0 \neq 1$, there is an axisymmetric solution. A nonaxisymmetric solution exists only for $\rho_0 < 1$ and has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{pmatrix}, \qquad \mathbf{u} = f_1(0,\ A_{13}x_1 + A_{33}x_3,\ -A_{22}x_2) \quad \text{outside} \quad S,$$

where $A_{22} > 0$, $A_{33} > 0$, and $A_{22} > A_{33}$ are arbitrary;

$$A_{11} = (A_{22} - A_{33})(1 + \frac{4}{1-\rho_0} A_{33}/A_{22}); \quad A_{13} = \pm 2A_{33}\sqrt{\frac{A_{22} - A_{33}}{(1-\rho_0)A_{22}}}; \qquad f_1 = \pm\omega_0\sqrt{\frac{1-\rho_0}{A_{22}(A_{22} - A_{33})}}.$$

4. Integrals of Motion. System (2.2) has the following integrals:

$$J_1 = \det A, \quad J_2 = \mathrm{sp}((\rho_0 B_0^* B_0^{\cdot} - B_1^* B_1)A^{-1}), \quad J_3 = (\rho_0 B_0 - B_1)A^{-1} - A^{-1}(\rho_0 B_0^* - B_1^*),$$
$$J_4 = \mathrm{sp}(((B_0^* - B_0)A^{-1})^2), \quad J_5 = \mathrm{sp}(((B_1^* - B_1)A^{-1})^2).$$

5. Solution of the Equations of Vortex Motion. When $\rho_0 = 1$, system (2.2) is integratable in elementary functions. Antisymmetric matrix F is an integral of (2.3). When $F = 0$, the solution is trivial: $B_0 = B_1$. It will therefore be assumed that $F \neq 0$. Since system (2.2) is invariant with respect to orthogonal transformations of the coordinate system, we direct the x_3 axis so that matrix F has the form

$$F = \begin{pmatrix} 0 & -2f & 0 \\ 2f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f \neq 0$$

(do not confuse with $f = \mathbf{x} \cdot A\mathbf{x} - 1$). We denote

$$B = B_0 - \frac{1}{2}FA = \frac{1}{2}(B_0 + B_1),$$

and write Eq. (2.4) as $BF - FB^* + qI = 0$. Writing it termwise, we obtain

$$B = \begin{pmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{11} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}, \quad q = 0. \tag{5.1}$$

Substituting $B_0 = B + (1/2)FA$ and $B_1 = B - (1/2)FA$ into the expression for q, we have $q = 0$. Therefore, the second equation of (5.1) is a consequence of the first. Substituting $B_0 = B + (1/2)FA$ into the second and third equations of (2.2) and taking into account that $BF = FB^*$ and $\mathrm{sp}B = 0$, we obtain a normalized system of equations of order 8 with respect to the unknown functions B_{13}, B_{23}, and A:

$$\dot{B} - \dot{B}^* + B^2 - B^{2*} + \frac{1}{4}(FA)^2 - \frac{1}{4}(FA)^{2*} = 0, \quad \dot{A} + AB + B^*A = 0. \tag{5.2}$$

This system contains an arbitrary function $B_{11}(t)$.

We transform to the coordinate system rotating around the x_3 axis. This means substitution of variables in Eqs. (5.2):

$$B \to B' = UBU^*, \qquad \dot{B} \to U\dot{B}U^* = \dot{B}' + PB' - B'P.$$

Here,

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad P = U\dot{U}^* = \begin{pmatrix} 0 & -\dot{\vartheta} & 0 \\ \dot{\vartheta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Matrix A is transformed in a similar manner. Matrix F does not change, inasmuch as it commutes with U. We choose U such that $B'_{23} = 0$. The equations for B' take the form

$$\dot{B}'_{13} - B'_{11}B'_{13} + f^2\Delta'_{31} = 0, \qquad \dot{\vartheta}B'_{13} + f^2\Delta'_{32} = 0,$$

where Δ'_{ij} is the cofactor of element A'_{ij} of matrix A'. Since the system does not contain the variable ϑ, its order decreases to 7. Recall that ϑ is the angle between the x'_1 axis and the immovable abscissa. It is remarkable that, by choosing an appropriate substitution of variables, one can eliminate the arbitrary function $B'_{11}(t)$. This substitution is equivalent to the following transformation of the coordinates, time, and B'_{13}:

$$x_i' \to h^{-1}x_i' \quad (i = 1, 2), \quad x_3' \to h^2 x_3', \quad t \to \int h^{-2}dt, \quad B_{13}' \to h^{-1}B_{13}' \quad \left(h = \exp\left(\int_0^t B_{11}'(\tau)d\tau\right)\right).$$

It is therefore sufficient to consider only the case of $B_{11}' = 0$. We write the resulting system of equations in the variables Δ_{ij}' (the primes are omitted):

$$2\dot{\omega} + f^2\Delta_{31} = 0, \quad 2\omega\dot{\vartheta} + f^2\Delta_{32} = 0, \quad \dot{\Delta}_{11} = 2\dot{\vartheta}\Delta_{21} + 4\omega\Delta_{31}, \quad \dot{\Delta}_{21} = \dot{\vartheta}(-\Delta_{11} + \Delta_{22}) + 2\omega\Delta_{32},$$
$$\dot{\Delta}_{22} = -2\dot{\vartheta}\Delta_{21}, \quad \dot{\Delta}_{31} = \dot{\vartheta}\Delta_{32} + 2\omega\Delta_{33}, \quad \dot{\Delta}_{32} = -\dot{\vartheta}\Delta_{31}, \quad \dot{\Delta}_{33} = 0. \qquad (5.3)$$

Here, $\omega = (1/2)B_{13}$. Note that Eq. (5.2) for A in the rotating coordinate system must be multiplied by A^{-1} both on the left and on the right. After this, one should use the rule of differentiation of an inverse matrix: $d(A^{-1})/dt = -A^{-1}\dot{A}A^{-1}$. Note next that $\Delta_{ij} = (A^{-1})_{ji}\Delta$ and $\Delta = \det A$ is an integral of motion. The variables Δ_{ij} have a clear geometrical meaning. The point of tangency of the ellipsoid and the plane $x_k = \text{const}$ has the coordinates $(\Delta\Delta_{kk})^{-1/2}$ $(\Delta_{k1}, \Delta_{k2}, \Delta_{k3})$. The projection of the ellipsoid onto the plane $x_3 = 0$ is bounded by the ellipse $\Delta_{22}x_1^2 + \Delta_{11}x_2^2 - 2\Delta_{21}x_1x_2 = (\Delta_{11}\Delta_{22} - \Delta_{21}^2)/\Delta$.

System (5.3) has the following integrals:

$$\gamma_1 = (\Delta_{11} + \Delta_{22})\Delta_{33} - \Delta_{31}^2 - \Delta_{32}^2, \quad \gamma_2 = \Delta_{33}, \quad \gamma_3 = \Delta_{11} + \Delta_{22} + 4f^{-2}\omega^2, \quad \gamma_4 = \omega\Delta_{32},$$
$$\gamma_5 = 4f^{-2}\omega^2\Delta_{22} + \Delta_{11}\Delta_{22} - \Delta_{21}^2, \quad \Delta^2 = (\Delta_{11}\Delta_{22} - \Delta_{21}^2)\Delta_{33} + 2\Delta_{21}\Delta_{31}\Delta_{32} - \Delta_{11}\Delta_{32}^2 - \Delta_{22}\Delta_{31}^2. \qquad (5.4)$$

It follows from integrals γ_1 and γ_2 that the plane section $x_3 = 0$ of the ellipsoid retains its shape and the ellipsoid constantly touches the immovable plane $x_3 = \sqrt{\Delta_{33}/\Delta}$.

All the variables in integrals (5.4) are easily expressed in terms of ω. The function ω can be found by means of the quadrature

$$-f^2t = \int \frac{d\omega^2}{\sqrt{\gamma_1'\omega^2 - 4f^{-2}\Delta_{33}\omega^4 - \gamma_4^2}} \qquad (\gamma_1' = \gamma_3\Delta_{33} - \gamma_1 > 0).$$

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